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JHEP05(2009)096

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Tension term, interchange symmetry, and the analogy of energy and tension laws of the AdS soliton solution

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ABSTRACT: In this paper, we reconsider the energy and tension laws of the Ricci flat black hole by taking the contribution of the tension term into account. After this considering and inspired by the interchange symmetry between the Ricci flat black hole and the AdS soliton solution which arises from the double analytic continuation of the time and compact spatial direction, we find out the analogy of the energy and tension laws of the AdS soliton solution. Moreover, we also investigate the energy and tension laws of the boosted Ricci flat black hole, and discuss the boosted AdS soliton solution. However, although there is the same interchange symmetry between the boosted Ricci flat black hole and boosted AdS soliton, the analogy of laws of the boosted AdS soliton solution may be of no sense for the existence of the closed timelike curves and conical singularity. In spite of that, the conserved charges such as the energy and momentum of the boosted AdS soliton are well-defined, and an interesting result is that its energy is lower than that of the static AdS soliton. On the other hand, note that although the laws obtained above are the same as those of the asymptotically flat case, the underlying deduced contents are different. Thus, our results could also be considered as a simple generalization to the asymptotically AdS case. Moreover, during the calculation, we find that there may be a new way to define the gravitational tension which can come from the quasi-local stress tensor of the counter-term method.

KEYWORDS: Black Holes in String Theory, AdS-CFT Correspondence, Classical Theories of Gravity, Black Holes

ARXIV EPRINT: [0904.1250](https://arxiv.org/abs/0904.1250)

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1 Introduction

It is well-known that the positive energy theorems ensure the energies of the solutions approaching AdS spacetime globally cannot be negative [1–3]. However, if the considering spacetimes are locally asymptotically AdS but not globally, the positive energy theorem may not hold. The Horowitz-Myers AdS soliton solution is this kind of particular solution [4]. This AdS soliton solution is important not only for its negative energy, but also for the agreement with the Casimir energy in the field theory viewed from the AdS/CFT correspondence [5]. Furthermore, it has also been found that there is a similar phase transition like the Hawking-Page phase transition between the Ricci flat black hole and the AdS soliton solution, and it could be connected with the confinement/deconfinement phase transition in QCD [6–8].

Although many properties of this AdS soliton solution have been studied, the analogy of its energy and tension laws is absent, and it is simply because its entropy is zero and the period of the imaginary time is arbitrary. Recently, inspired from the interchange symmetry between the KK bubble and the corresponding black hole which are all asymptotically flat, D.Kastor et al obtained some interesting results of the KK bubble after defining some new quantities such as its surface gravity and the area of the KK bubble [10](Note that the surface gravity here is associated with the spacelike Killing field which translates around the compact spatial coordinate, and more details can be found in [11]). In our paper, viewed from the similar interchange symmetry between the AdS soliton solution and the Ricci flat black hole, we first reinvestigate the energy and tension laws of the Ricci flat black hole by considering the contribution of the tension term [9, 10, 13–15], then we investigate the analogy of the laws of the AdS soliton solution. We find the same analogy as that of the laws of the KK bubble. In addition, we also investigate the laws of the boosted Ricci

flat black hole [12]. The boosted Ricci flat black hole can be obtained from the static Ricci flat black hole by a boost transformation along the compact spatial coordinate [18]. Note that, because the spatial coordinate is compact, the boosted Ricci flat black hole is not equivalent to the static one globally [16–18]. And these kind of globally stationary but locally static spacetimes could be considered as the gravitational analog of the Aharonow-Bohm effect [19, 20]. Similarly, for the static AdS soliton solution, we can also make a boost transformation along the compact spatial coordinate of the static AdS soliton solution, and then obtain the boosted AdS soliton solution. Like the AdS soliton solution, the boosted AdS soliton solution also has the same interchange symmetry with the above boosted Ricci flat black hole. However, there are closed timelike curves and conical singularity in the boosted AdS soliton solution, thus this solution is ill in physics and the direct analogy of the energy and tension laws of the boosted AdS soliton solution is of no sense. In spite of that, its conserved charges such as the energy and momentum are well-defined, and an interesting result is that the energy of the boosted AdS soliton is lower than that of the AdS soliton. On the other hand, note that although here we can easily find that the energy and tension laws of the boosted Ricci flat black hole or the AdS soliton solution are the same as those of the asymptotically flat case, the underlying contents are not the same. First of all, the methods of calculating the conserved charges are different. Because what they discuss are the asymptotically flat cases, the well-known ADM calculation can be used in their cases [10, 12]. However, it is invalid and there have been several methods to calculate the conserved charges in the asymptotically AdS case [21–26]. Here we just use the surface counterterm method or Euclidean method. Second, they obtain the laws by using the Hamiltonian perturbation theory techniques [14, 27], and more expressive is that they should use the Hamiltonian formalism presented by the ADM method [10, 12]. While for black holes we obtain the laws just by applying the Euclidean method [22], and basing on this we obtain the laws of AdS soliton by using the property of interchange symmetry. During the derivation of laws, we do not need the explicit formalisms of conserved charges. Thus, our results could also be considered as a simple generalization of the results in asymptotically flat case to the asymptotically AdS case [10, 12].

The rest of paper is organized as follows. In section II, we reinvestigate the energy and tension laws of the Ricci flat black hole by considering the contribution of the tension term. In section III, inspired from the interchange symmetry with the Ricci flat black hole, we obtain the analogy of the laws of the AdS soliton solution. In section IV, we generalize the above discussion in section II to the case of the boosted Ricci flat black hole. In section V, we consider the boosted AdS soliton solution. Finally, in section VI, we give a brief conclusion and discussion.

2 Reinvestigation of the energy and tension laws of the Ricci flat black hole

The so-called Ricci flat black hole solution considered here is [4, 5]

$$ds^2 = \frac{r^2}{l^2} \left[- \left(1 - \frac{r_0^4}{r^4} \right) dt^2 + dy^2 + (dx^i)^2 \right] + \left(1 - \frac{r_0^4}{r^4} \right)^{-1} \frac{l^2}{r^2} dr^2. \quad (i = 1, 2) \quad (2.1)$$

which arises in the near-horizon geometry of p-brane and is asymptotically the five dimensional AdS metric. It is easy to find that its event horizon locates at $r_+ = r_0$. And in order to remove the conical singularity at the horizon, the Euclidean time τ must have a period $\beta = \frac{\pi l^2}{r_0}$. Note that, the coordinate y is a compact spatial coordinate, and its period is η . As the usual treatment, we can use the Euclidean method to research the thermodynamics of the Ricci flat black hole [22]. Choosing the pure AdS spacetime as the reference background, we can easily obtain the Euclidean action of the Ricci flat black hole

$$I_E = -\frac{\beta r_0^4}{16\pi l^5} \eta V_2. \tag{2.2}$$

where V_2 is the coordinate volume of the surfaces parameterized by x^i . Thus, the free energy of the Ricci flat black hole evaluated on the pure AdS background is [22]

$$F \equiv \frac{I_E}{\beta} = E - TS = -\frac{r_0^4}{16\pi l^5} \eta V_2. \tag{2.3}$$

And the energy and entropy are

$$E = \frac{\partial I_E}{\partial \beta} = \frac{3r_0^4}{16\pi l^5} \eta V_2, \tag{2.4}$$

$$S = \beta \frac{\partial I_E}{\partial \beta} - I_E = \frac{\eta V_2 r_0^3}{4l^3}. \tag{2.5}$$

From (2.5), it can be seen that the entropy S is exactly equal to 1/4 of the horizon area A , which implicates that those thermodynamical equations hold

$$dF = -SdT, dE = TdS. \tag{2.6}$$

It should be emphasized that we have not considered the contribution of tension term to the laws above, i.e gravitational tension. And it is known that the gravitational tension term could contribute to the first law in the case of the black p-branes or black string if the size of the compact spatial coordinate is allowed to be changed. The fact is that the geometry looks locally like the black string when is far from the horizon of the Ricci flat black hole, thus the gravitational tension term may also contribute to the thermodynamical laws [13]. And it is true that if assuming the free energy in (2.3) is also the function of η , we can obtain not only the energy and entropy but also the gravitational tension

$$\begin{aligned} E &= \left(\frac{\partial I_E}{\partial \beta} \right)_\eta = \frac{3r_0^4}{16\pi l^5} \eta V_2, \\ S &= \beta \left(\frac{\partial I_E}{\partial \beta} \right)_\eta - I_E = \frac{\eta V_2 r_0^3}{4l^3}, \\ \Gamma &= \frac{1}{\beta} \left(\frac{\partial I_E}{\partial \eta} \right)_\beta = -\frac{r_0^4}{16\pi l^5} V_2. \end{aligned} \tag{2.7}$$

On the other hand, in a $d + 1$ dimensional spacetime \mathcal{M} , the conserved charge associated with the killing vector ξ^μ generating an isometry of the boundary geometry $\partial\mathcal{M}$ defined through the quasilocal stress tensor is [21, 25]

$$Q_\xi = \int_\Sigma d^{d-1}x \sqrt{\sigma} (u^\mu T_{\mu\nu} \xi^\nu). \tag{2.8}$$

where Σ is a spacelike hypersurface in the boundary $\partial\mathcal{M}$, and u^μ is the timelike unit vector normal to it. σ_{ab} is the metric on Σ defined as

$$\gamma_{\mu\nu}dx^\mu dx^\nu = -N_\Sigma^2 dt^2 + \sigma_{ab}(dx^a + N_\Sigma^a dt)(dx^b + N_\Sigma^b dt) \quad (2.9)$$

and $\gamma_{\mu\nu}$ is the metric on the boundary. Thus, the energy related with the timelike killing vector ξ^μ and the momentum could be defined respectively as

$$E = \int_\Sigma d^{d-1}x \sqrt{\sigma} N_\Sigma (u^\mu T_{\mu\nu} u^\nu), \quad (2.10)$$

$$P_a = \int_\Sigma d^{d-1}x \sqrt{\sigma} \sigma_{ab} u_\mu T^{b\mu}. \quad (2.11)$$

According to the surface counterterm method, the quasilocal stress tensor for the asymptotically AdS_5 solution is [21]

$$T_{\mu\nu} = \frac{1}{8\pi} \left(\theta_{\mu\nu} - \theta \gamma_{\mu\nu} - \frac{3}{l} \gamma_{\mu\nu} - G_{\mu\nu} \right). \quad (2.12)$$

where all the above tensors refer to the boundary metric $\gamma_{\mu\nu}$ defined on the hypersurface $r = constant$, and $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R\gamma_{\mu\nu}$ is the Einstein tensor of $\gamma_{\mu\nu}$, $\theta_{\mu\nu} = -\frac{1}{2}(\nabla_\mu n_\nu + \nabla_\nu n_\mu)$ is the extrinsic curvature of the boundary with the normal vector n^μ in the spacetime. Therefore, we can easily obtain the useful quasi-local stress tensor of the Ricci flat black hole (2.1)

$$8\pi T_{tt} = \frac{3r_0^4}{2l^3 r^2} + \dots \quad (2.13)$$

And the energy is

$$E = \frac{3r_0^4}{16\pi l^5} \eta V_2. \quad (2.14)$$

which is consistent with the above result in (2.7). In addition, the general definition of gravitational tension in a given asymptotically translationally-invariant spatial direction (i.e. x) of a D dimensional space-time is [9]

$$\Gamma = \frac{1}{\Delta t} \frac{1}{8\pi} \int_{S_x^\infty} \left[F(K^{(D-2)} - K_0^{(D-2)}) - F^v p_{\mu\nu} r^\nu \right] \quad (2.15)$$

here $S_x^\infty = \Sigma_x \cap \Sigma^\infty$ and Σ_x is the hypersurface $x = const$ with unit normal vector n^μ , and Σ^∞ is the asymptotic boundary of the spacetime with unit normal vector r^μ . The space-like killing vector X^μ corresponding to the translationally-invariant spatial direction x is decomposed into normal and tangential parts to Σ_x that

$$X^\mu = F n^\mu + F^\mu \quad (2.16)$$

and the extrinsic curvature tensor on Σ_x with respect to n^μ is $K_{\mu\nu}$, while $K^{(D-2)}$ is the extrinsic curvature of the surface S_x^∞ in Σ_x , and $K_0^{(D-2)}$ is the corresponding extrinsic curvature of the surface S_x^∞ in the reference space $(M, (g_0)_{\mu\nu})$. The metric with respect to n^μ on Σ_x is

$$h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu \quad (2.17)$$

while the corresponding canonical momentum $p_{\mu\nu}$ with respect to $h_{\mu\nu}$ is

$$p^{\mu\nu} = \frac{1}{\sqrt{h}}\pi^{\mu\nu} = K^{\mu\nu} - Kh^{\mu\nu} \quad (2.18)$$

Thus, from this general definition of gravitational tension (2.15), we can obtain the gravitational tension along the compact spatial direction y in Ricci flat black hole (2.1)

$$\Gamma = -\frac{r_0^4}{16\pi l^5}V_2. \quad (2.19)$$

which is also consistent with the above result in (2.7). And these consistences of energy, tension and entropy implicate that after adding the contribution of tension term the first laws in (2.6) are

$$\begin{aligned} dF &= -SdT + \Gamma d\eta = -\frac{1}{8\pi}A_H d\kappa_H + \Gamma d\eta, \\ dE &= TdS + \Gamma d\eta = \frac{1}{8\pi}\kappa_H dA_H + \Gamma d\eta. \end{aligned} \quad (2.20)$$

where $T = 1/\beta = \kappa_H/2\pi$ and $S = A_H/4$. Using these conserved charges, we can also easily check that

$$E - TS = \Gamma\eta. \quad (2.21)$$

which is very similar with the Smarr relation. Thus from (2.21) and (2.20), we can obtain the tension law that

$$\eta d\Gamma = -SdT. \quad (2.22)$$

which can be found to have the same formalism with the static Kaluza-Klein black hole which is asymptotically flat in Refs [10, 12].

3 The AdS soliton solution, interchange symmetry, and analogy of energy and tension laws

The AdS soliton solution is [4]

$$ds^2 = \frac{r^2}{l^2} \left[\left(1 - \frac{r_0^4}{r^4}\right) dy^2 - dt^2 + (dx^i)^2 \right] + \left(1 - \frac{r_0^4}{r^4}\right)^{-1} \frac{l^2}{r^2} dr^2. \quad (i = 1, 2) \quad (3.1)$$

with the coordinate r restricted to $r \geq r_0$. Again, the coordinate y could be identified with period $\eta = \frac{\pi l^2}{r_0}$ to avoid a conical singularity at $r = r_0$. Note that this spacetime is completely nonsingular and globally static. And it can be obtained from the Ricci flat black hole metric (2.1) with the double analytic continuation such that

$$t \rightarrow iy, y \rightarrow it. \quad (3.2)$$

which arises an interesting interchange symmetry between the AdS soliton solution and the Ricci flat black hole.

Using the same surface counterterm method, we can calculate the useful quasilocal stress tension of AdS soliton [21]

$$8\pi T_{tt} = -\frac{r_0^4}{2l^3 r^2} + \dots \quad (3.3)$$

Thus, the energy is

$$E = -\frac{r_0^4}{16\pi l^5} \eta V_2. \quad (3.4)$$

In addition, the tension of the AdS soliton (along the direction of the compact coordinate y) from the general definition (2.15) is

$$\Gamma = \frac{3r_0^4}{16\pi l^5} V_2. \quad (3.5)$$

Eqs. (3.4), (3.5) can explicitly manifest the interchange symmetry with the Ricci flat black hole compared with its energy and tension.

In the above section, during deriving the laws of Ricci flat black hole, we mainly base on an underlying assumption that the equations (2.20) hold, and then find the consistence with the calculations by other methods. However, for the AdS soliton solution, the first problem is these equations may not hold because the period of the imaginary time which usually relates with the temperature is arbitrary in the AdS soliton spacetime. Moreover, if we take the entropy just as the usual Bekenstein-Hawking entropy (it is the 1/4 of the horizon area), we could find the entropy is zero. Thus, the direct analogy of the mass and tension laws of the AdS soliton solution like (2.20), (2.22) seems to be absent. In spite of that, inspired from the interchange symmetry between the black hole and AdS soliton, it may have the analogy. And it is true that it has been found the similar analogy of the KK bubble in ref. [10] where it discusses the asymptotically flat case. As same as that of KK bubble, we can also first define some new quantities, such as the surface gravity and the area of the AdS soliton. And according to these definitions, the surface gravity and the area of the AdS soliton are [10]

$$\kappa_s = \frac{2r_0}{l^2}, \quad A_s = \frac{V_2 r_0^3}{l^3}. \quad (3.6)$$

However, here we would not use the Hamiltonian perturbation techniques to deduce the laws of AdS soliton until one finds its appropriate formalisms of the conserved charges and gravitational tension as those of KK bubble. And we just base on its interchange symmetry with the Ricci flat black hole (2.1). From the quantities in (3.6) and those in (3.4), (3.5), we can make an easy displacement in (2.20), (2.21) and (2.22) by using the interchange symmetry such that

$$E \rightarrow \Gamma \eta, \quad T \rightarrow T, S \rightarrow S \eta, \quad \Gamma \rightarrow E/\eta. \quad (3.7)$$

Thus we can obtain the reduced relations

$$d\Gamma = \frac{1}{8\pi G} \kappa_s dA_s. \quad (3.8)$$

$$dE = -\frac{1}{8\pi G} \eta A_s d\kappa_s + \left(\Gamma - \frac{1}{8\pi G} \kappa_s A_s \right) d\eta. \quad (3.9)$$

From which, we can easily check out that they hold by using the quantities in (3.4), (3.5), (3.6) and see that they have the similar formalisms with the laws of black hole. Thus they can be naturally considered as the analogy of the energy and tension laws of the AdS soliton. The most interesting thing is that they has the same formalism with the result of the K-K bubble in Ref [10] where it is deduced by using the Hamiltonian perturbation theory techniques [14, 27]. Thus, it is more convincible that they could be considered as the analogy of the energy and tension laws of the AdS soliton.

4 The energy and tension laws for boosted Ricci flat black hole

The boosted Ricci flat black hole can be obtained from (2.1) by the following boost transformation [18]

$$\begin{aligned} t &\rightarrow t \cosh \alpha - y \sinh \alpha, \\ y &\rightarrow -t \sinh \alpha + y \cosh \alpha. \end{aligned} \quad (4.1)$$

where α is the boost parameter and the boost velocity is $v = \tanh \alpha$. Thus, the metric of the boosted Ricci flat black hole is

$$ds^2 = \frac{r^2}{l^2} \left[-dt^2 + dy^2 + \frac{r_0^4}{r^4} (dt \cosh \alpha - dy \sinh \alpha)^2 + (dx^i)^2 \right] + \left(1 - \frac{r_0^4}{r^4} \right)^{-1} \frac{l^2}{r^2} dr^2. \quad (i = 1, 2) \quad (4.2)$$

Note that, because the coordinate y in (4.1) is periodic, the solution (4.2) is not equivalent to the static Ricci flat black hole (2.1) globally [16–18]. And in order to remove the conical singularity at the horizon $r = r_0$, the Euclidean time τ in (4.2) could have a period $\beta = \frac{\pi l^2 \cosh \alpha}{r_0}$. Following the same procedure as section II, at first we do not consider the contribution from the gravitational tension term in the laws of thermodynamics. After choosing the pure AdS spacetime as the background and using the same Euclidean method, we can obtain the Euclidean action of the boosted Ricci black hole to be [22]

$$I_E = -\frac{\beta r_0^4}{16\pi G l^5} \eta V_2. \quad (4.3)$$

Note that, although the Euclidean action is the same as that of the static Ricci flat black hole (2.2), the relationship between β and r_0 is different. Moreover, here the thermal function related with Euclidean action is the Gibbons free energy [22]

$$G \equiv \frac{I_E}{\beta} = E - TS - vP. \quad (4.4)$$

and G is the function of not only β but also the boost velocity v . Thus, the energy, entropy and momentum are

$$\begin{aligned} E &= \left(\frac{\partial I_E}{\partial \beta} \right)_v - \frac{v}{\beta} \left(\frac{\partial I_E}{\partial v} \right)_\beta = \frac{(3 + 4a^2)r_0^4}{16\pi l^5} \eta V_2, \\ S &= \beta \left(\frac{\partial I_E}{\partial \beta} \right)_v - I_E = \frac{\eta V_2 r_0^3}{4l^3} \sqrt{1 + a^2}, \\ P &= -\frac{1}{\beta} \left(\frac{\partial I_E}{\partial v} \right)_\beta = \frac{\eta V_2 r_0^4}{4\pi l^5} a \sqrt{1 + a^2}. \end{aligned} \quad (4.5)$$

where $a \equiv \sinh \alpha$ and it could be easily seen that the entropy S is also exactly equal to $1/4$ of the horizon area A , which implicates that the following relations hold

$$\begin{aligned} dG &= -SdT - Pdv, \\ dE &= TdS + vdP. \end{aligned} \quad (4.6)$$

Again if assuming the Gibbons free energy G in (4.4) is also the function of η , we can also obtain the gravitational tension

$$\begin{aligned} E &= \left(\frac{\partial I_E}{\partial \beta} \right)_{v,\eta} - \frac{v}{\beta} \left(\frac{\partial I_E}{\partial v} \right)_{\beta,\eta} = \frac{(3 + 4a^2)r_0^4}{16\pi l^5} \eta V_2, \\ S &= \beta \left(\frac{\partial I_E}{\partial \beta} \right)_{v,\eta} - I_E = \frac{\eta V_2 r_0^3}{4l^3} \sqrt{1 + a^2}, \\ P &= -\frac{1}{\beta} \left(\frac{\partial I_E}{\partial v} \right)_{\beta,\eta} = \frac{\eta V_2 r_0^4}{4\pi l^5} a \sqrt{1 + a^2}, \\ \Gamma &= \frac{1}{\beta} \left(\frac{\partial I_E}{\partial \eta} \right)_{\beta,v} = -\frac{r_0^4}{16\pi l^5} V_2. \end{aligned} \quad (4.7)$$

On the other hand, according to the definition, the useful quasi-local stress tensor of the boosted Ricci flat black hole (4.2) is [21]

$$\begin{aligned} 8\pi T_{tt} &= \frac{(3 + 4 \sinh^2 \alpha) r_0^4}{2l^3 r^2} + \dots \\ 8\pi T_{ty} &= -\frac{2 \sinh \alpha \cosh \alpha r_0^4}{l^3 r^2} + \dots \end{aligned} \quad (4.8)$$

From which the energy and momentum can be calculated to be

$$\begin{aligned} E &= \frac{(3 + 4 \sinh^2 \alpha) r_0^4}{16\pi l^5} \eta V_2, \\ P &= \frac{\sinh \alpha \cosh \alpha r_0^4}{4\pi l^5} \eta V_2. \end{aligned} \quad (4.9)$$

where the energy and momentum are consistent with the above results in (4.7). And this consistence could implicate that after adding the contribution of tension term the first laws in (4.6) are

$$\begin{aligned} dG &= -SdT - Pdv + \Gamma d\eta, \\ dE &= TdS + vdP + \Gamma d\eta. \end{aligned} \quad (4.10)$$

However, if we use the general definition of gravitational tension (2.15), we can obtain the tension

$$\Gamma' = -\frac{(1 + 4 \sinh^2 \alpha) r_0^4}{16\pi l^5} V_2. \quad (4.11)$$

which is not consistent with the result in (4.7). Note that, this difference has also been found by D. Kastor et al, and they argued that the tension obtained in (4.7) was in fact an effective tension which was related to the general tension such that [12]

$$\Gamma = \Gamma' + \frac{vP}{\eta}. \quad (4.12)$$

From which, we can also find that when the boosted velocity is zero, the general tension is just equal to the effective tension.

Using these quantities in (4.7), (4.9), we can also check that

$$E - TS - vP = \Gamma\eta \tag{4.13}$$

Thus, from this relation (4.13) and the first energy law (4.10), the first tension law of boosted Ricci flat black hole is

$$SdT + Pdv + \eta d\Gamma = 0 \tag{4.14}$$

5 The boosted AdS soliton solution

Naturally, we can also make a boost transformation (4.1) along the compact coordinate y in the static AdS soliton solution (3.1). Thus, the boosted AdS soliton solution is

$$ds^2 = \frac{r^2}{l^2} \left[-dt^2 + dy^2 - \frac{r_0^4}{r^4} (dy \cosh \alpha - dt \sinh \alpha)^2 + (dx^i)^2 \right] + \left(1 - \frac{r_0^4}{r^4} \right)^{-1} \frac{l^2}{r^2} dr^2. \quad (i = 1, 2) \tag{5.1}$$

Note that this solution is also different from the static AdS soliton globally, and it is easy to see that the coordinate y in the boost transformation is compact. In addition, an interesting result is that this boosted AdS soliton solution also has the same interchange symmetry with the boosted Ricci flat black hole. That is, it can also be obtained from the boosted Ricci flat black hole solution by the double analytic continuation between the time and the compact coordinate y in (4.2). In the above section we have obtained the analogy of the energy and tension laws of the static AdS soliton solution through the inspiration from the interchange symmetry with the Ricci flat black hole. However, it's easily found that there are closed timelike curves in the boosted AdS soliton solution (5.1). Moreover, viewed from the physical point, after boosting along the compact coordinate y in static AdS soliton (3.1), the period of y would be shrunk to $\gamma^{-1}\eta$ where $\gamma = (1 - v^2)^{-1/2} = \cosh \alpha$ is the shrinking factor. However, the new period could not avoid the conical singularity. Thus, this boosted AdS soliton solution is ill in physics and the direct analogy of laws is of no sense. In spite of that, the conserved charges such as energy and momentum are well defined because they just depend on the properties of its asymptotic behavior. And the corresponding quasi-local stress tensor of the boosted AdS soliton solution can be obtained [21]

$$\begin{aligned} 8\pi T_{tt} &= -\frac{(1 + 4 \sinh^2 \alpha)r_0^4}{2l^3 r^2} + \dots \\ 8\pi T_{ty} &= \frac{2 \sinh \alpha \cosh \alpha r_0^4}{l^3 r^2} + \dots \end{aligned} \tag{5.2}$$

Thus, the energy and momentum of the boosted AdS soliton solution are

$$\begin{aligned} E &= -\frac{(1 + 4 \sinh^2 \alpha)r_0^4}{16\pi l^5} \eta V_2, \\ P &= -\frac{\sinh \alpha \cosh \alpha r_0^4}{4\pi l^5} \eta V_2. \end{aligned} \tag{5.3}$$

In addition, the general tension can also be obtained from the definition (2.15)

$$\Gamma = \frac{(3 + 4 \sinh^2 \alpha) r_0^4}{16\pi l^5} V_2. \tag{5.4}$$

These quantities in (5.3), (5.4) could explicitly manifest the interchange symmetry with the boosted Ricci flat black hole, too.

6 Conclusion and discussion

One of the motivations of this paper is to obtain the analogy of the energy and tension laws of the AdS soliton solution, which can give more understanding of this solution. In order to obtain them, we first reconsider the laws of the Ricci flat black hole by taking the contribution of the tension term into account. Then, inspired from the interchange symmetry between the Ricci flat black hole and AdS soliton, we finally obtain the analogy. In spite of that, how to understand the analogy of laws of the AdS soliton is an open question. Particularly, whether there is some underlying physical interpretations such as thermodynamical effects in it is worthy of further discussion. In addition, as a more general asymptotically AdS black hole solution, we also take the boosted Ricci flat black hole for example to give a simple generalization of the works by D.Kastor to the asymptotically AdS case. Note that, although here our formalisms of the laws of black holes or the static soliton are the same as those of the asymptotically flat cases, the underlying deduced contents are different. In principle, if we find the appropriate formalisms of conserved charges and gravitational tension, we perhaps can also use the Hamiltonian perturbation method to deduce these laws directly. And this possibility will be considered in the future work. As the corresponding solution which has the interchange symmetry with boosted Ricci flat black hole, we also consider the boosted AdS soliton solution. However, although there is the same interchange symmetry, this boosted AdS soliton solution is ill in physics because of the existence of the closed timelike curves and conical singularity. Thus, the direct analogy of energy and tension laws are of no sense. In spite of that, an interesting result is that the conserved charges such as the energy and momentum are well-defined for the boosted AdS soliton solution. Moreover, as we expected, its energy is smaller than that of the static AdS soliton solution. Thus, whether it can be considered as a violation case to the new positive energy conjecture proposed by G.T Horowitz and R.C Myers and how to understand it from the viewpoint of the AdS/CFT correspondence would also be interesting things to give further discussions. In addition, during calculating the conserved charges, we also find that perhaps there is a new way to define the gravitational tension from the quasi-local stress tensor defined in (2.12), because the gravitational tension can be easily found to be related to the corresponding stress tensor T_{yy} such that

$$\begin{aligned} \text{Ricci flat black hole : } \Gamma &= -\frac{r_0^4}{16\pi l^5} V_2, \quad T_{yy} = \frac{r_0^4}{16\pi l^3 r^2} + \dots \\ \text{Static AdS soliton : } \Gamma &= \frac{3r_0^4}{16\pi l^5} V_2, \quad T_{yy} = -\frac{3r_0^4}{16\pi l^3 r^2} + \dots \\ \text{Boosted Ricci flat black hole : } \Gamma &= -\frac{(1 + 4a^2)r_0^4}{16\pi l^5} V_2, \quad T_{yy} = \frac{(1 + 4a^2)r_0^4}{16\pi l^3 r^2} + \dots \end{aligned}$$

$$\text{Boosted AdS soliton} : \Gamma = \frac{3r_0^4}{16\pi l^5} V_2, \quad T_{yy} = -\frac{(3+4a^2)r_0^4}{16\pi l^3 r^2} + \dots \quad (6.1)$$

On the other hand, viewed from the physical interpretation of the stress tensor, its spatial diagonal components are related with the pressure, thus it is more convincing that there is a new possibility to define the gravitational tension. In fact, considering the interchange symmetry and the formalisms in (2.10), (2.15), we can give the new definition of the gravitational tension through the counterterm in the asymptotical AdS case that

$$\Gamma = -\frac{1}{\Delta t} \int_{S_\infty} d^{d-1}x \sqrt{\sigma} F(n^\mu T_{\mu\nu} n^\nu) \quad (6.2)$$

which can be easily checked that this new definition is satisfied in our cases.

Note that, after our paper appeared, Dr. Cristian Stelea showed me that they had also already given an exact formalism of the gravitational tension through the counterterm in their cases. Thus giving a more general rigorous definition of the gravitational tension through the counterterm is an open interesting question, and perhaps some clues could be found in their works [28].

Acknowledgments

Y.P Hu thanks Professor Rong-Gen Cai and Dr.Li-Ming Cao, Jia-Rui Sun, Xue-Fei Gong and Chang-Yong Liu for their helpful discussions. And Y.P Hu also thanks Dr. Cristian Stelea for his useful information. This work is supported partially by grants from NSFC, China (No. 10325525, No. 90403029 and No.10773002), and a grant from the Chinese Academy of Sciences.

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